Fairing of B-spline & Bézier-spline Surfaces

① Local Energy Fairing of B-spline Surfaces

② Extension to Bézier-spline Surfaces

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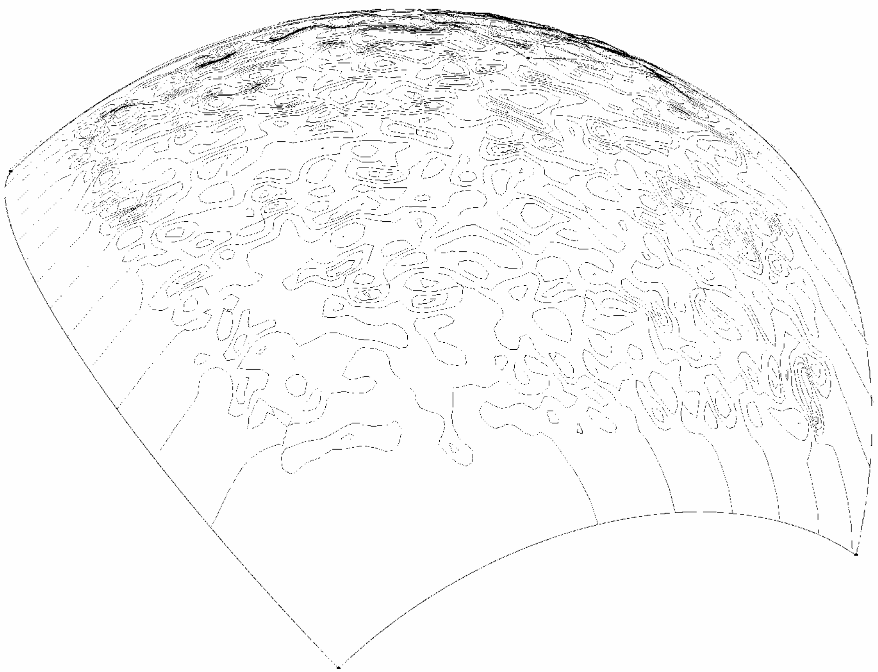
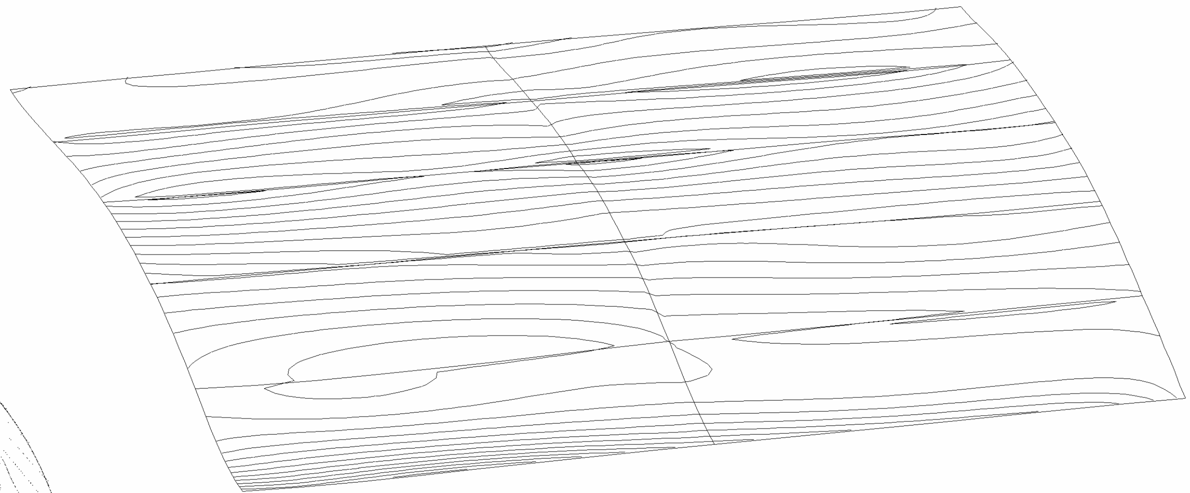
Problems (1)

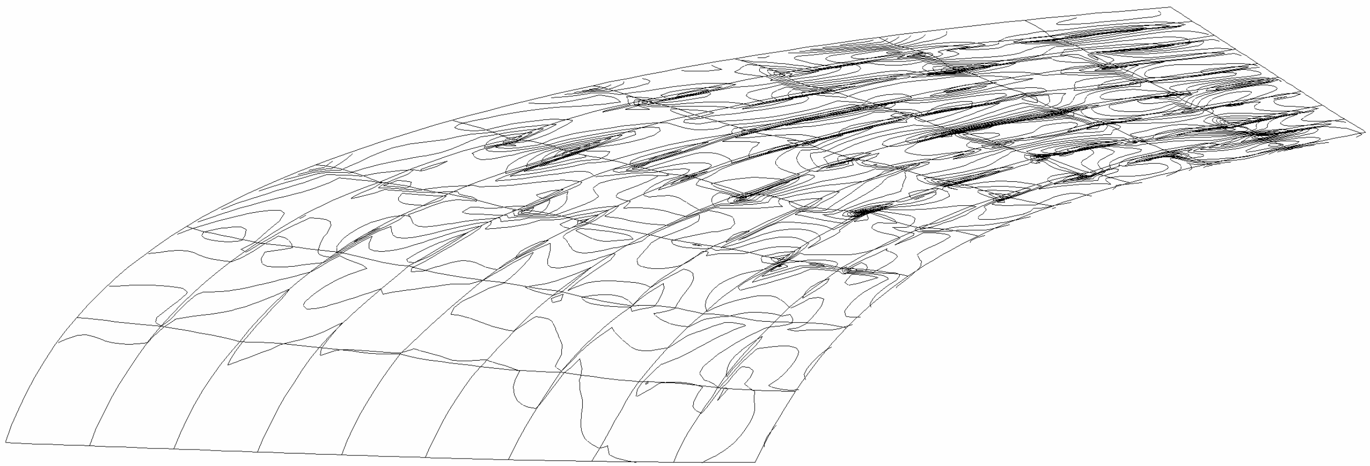
disturbed spherical surface



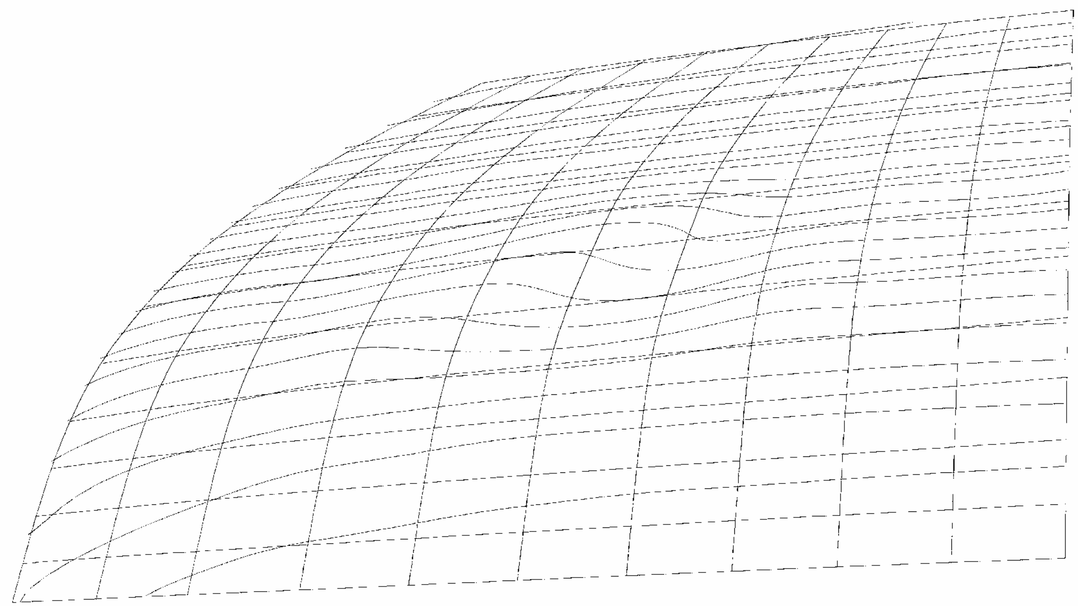


2 × 5 Bézier surfaces

Problems (2)

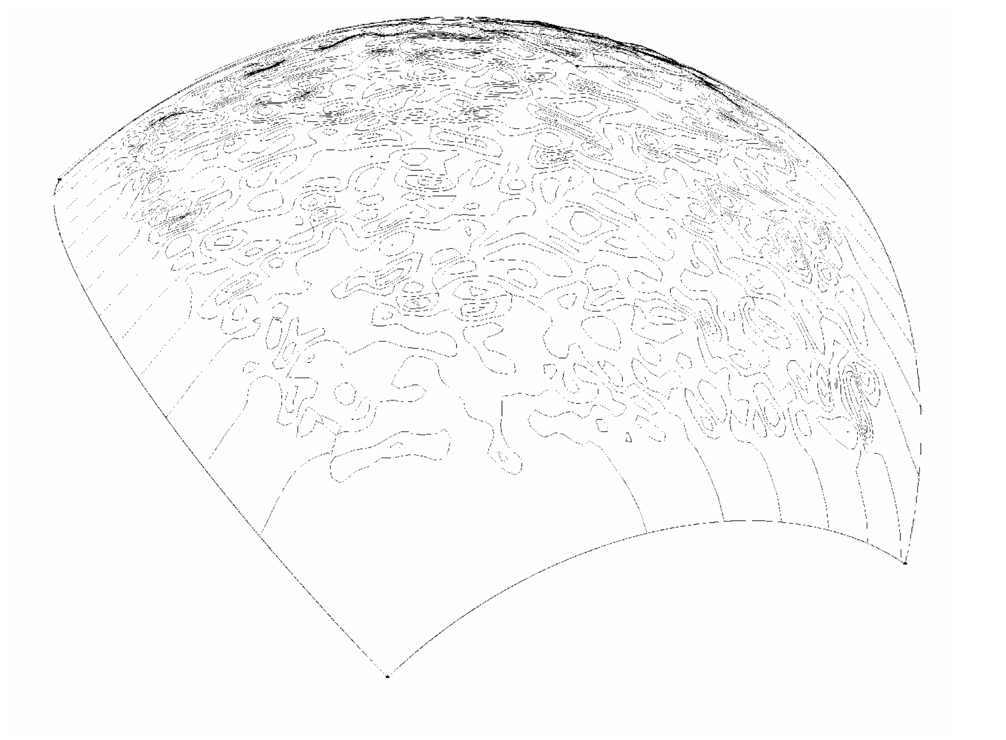


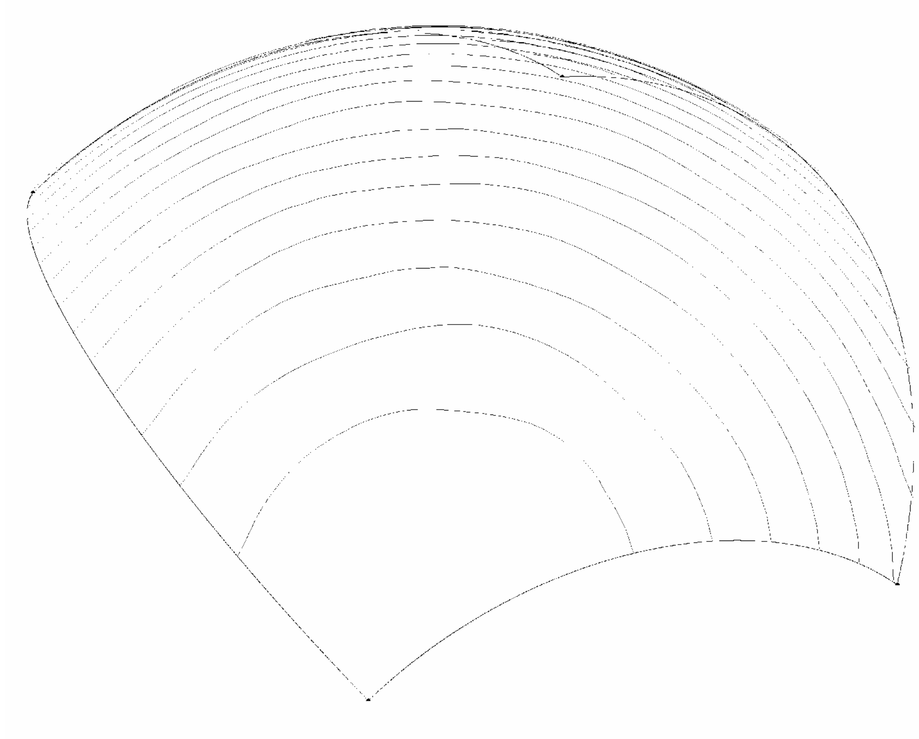
part of a hood



disturbed fender

One solution

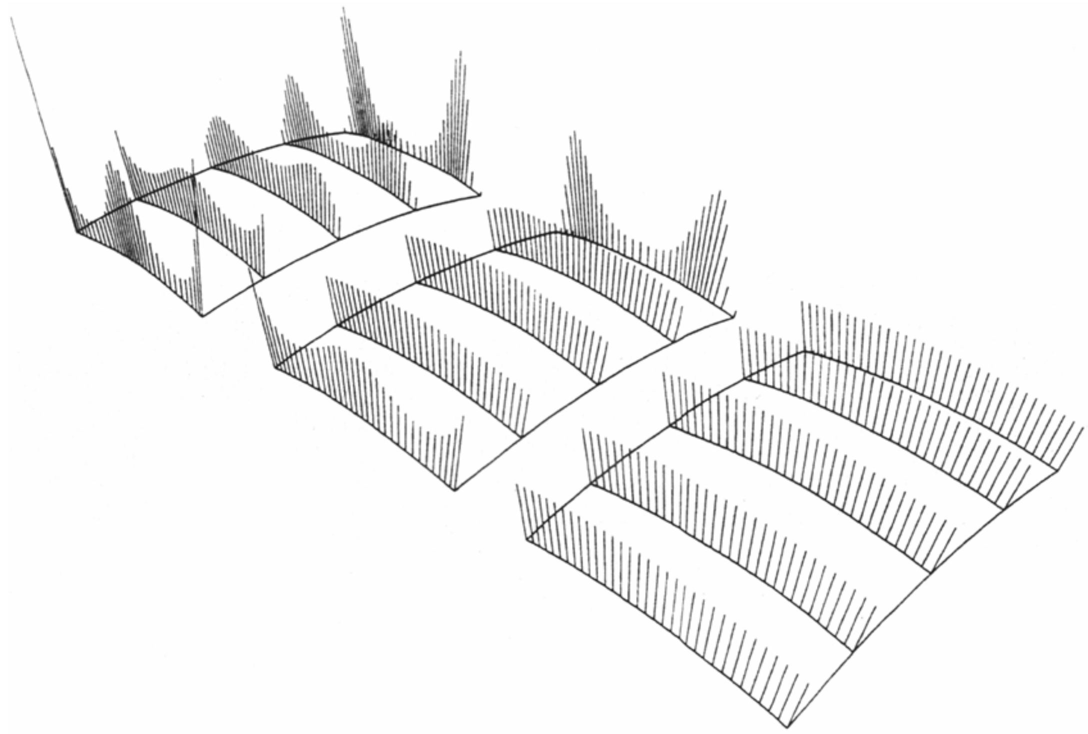




Local energy fairing of B-spline surfaces

[Hohenberger & Reuding, CAGD '95]

- Handle rows and columns of a surface like curves ?



- Here: fairing surfaces in a different way

Idea of the method

- Use a quadratic functional as the fairness criterion
- Local scheme
- Changing only one control point in every step
- Fulfilling a distance tolerance

Fairness criterion

Thin-plate energy [Courant & Hilbert ´31]

$$\Pi_P = \iint_S a\left(\kappa_1^2 + \kappa_2^2\right) + 2(1-b)\kappa_1 \cdot \kappa_2 \, dS$$

Linearization ↓ $(and\ \ a = b = 1)$

$$\Pi = \iint_A X_{uu}^2 + 2 \cdot X_{uv}^2 + X_{vv}^2 \, du\, dv$$

What is smooth?

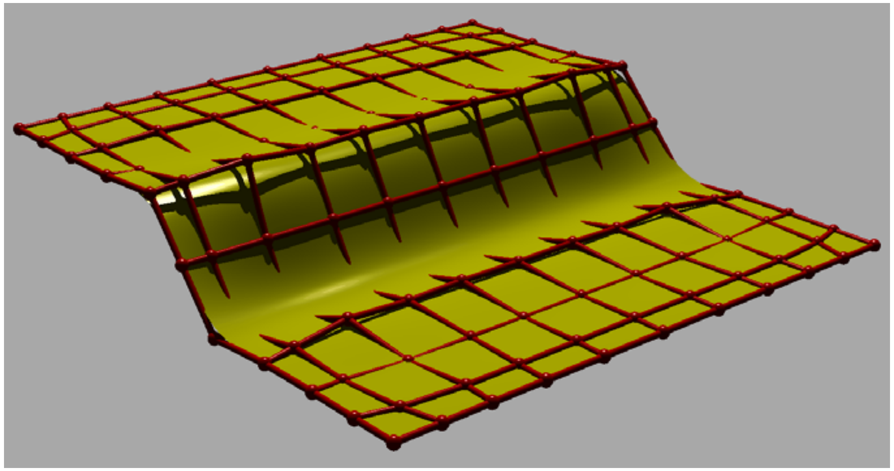
- Given ("unfair") B-spline surface $X(u,v)$
- Searching for a smooth B-spline surface $\tilde{X}(u,v)$

$$\Pi = \iint_A \tilde{X}_{uu}^2 + 2 \cdot \tilde{X}_{uv}^2 + \tilde{X}_{vv}^2 \, du \, dv \rightarrow \min_{\{\tilde{d}_{ij}\}}$$

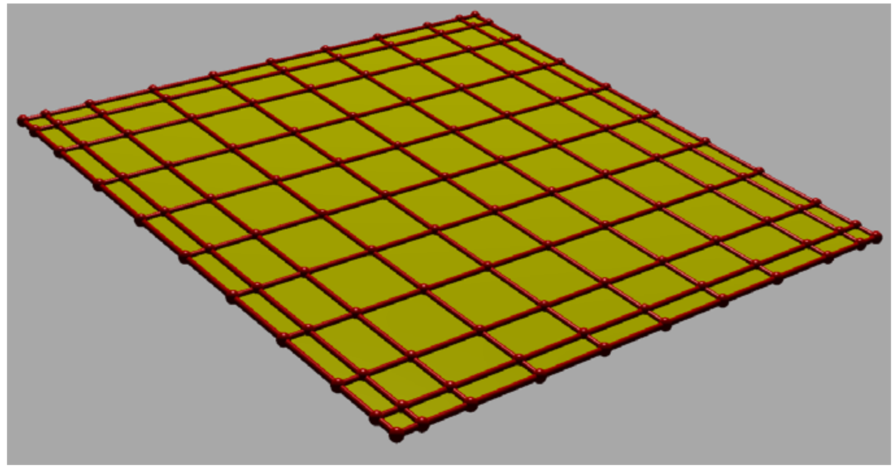
with $dist(\tilde{X}, X) \leq \varepsilon$

given distance tolerance

Distance tolerance



$\overset{?}{\rightarrow}$



$$dist(\tilde{X}, X) := \max_{u,v} \left\| X(u,v) - \tilde{X}(u,v) \right\|$$

too difficult

$$\leq \max \left\| d_{ij} - \tilde{d}_{ij} \right\|$$

simplier and a good bound

for low degrees

Problem

$$\Pi = \iint_A \tilde{X}_{uu}^2 + 2\cdot\tilde{X}_{uv}^2 + \tilde{X}_{vv}^2 \, du\, dv \rightarrow \min_{\{\tilde{d}_{ij}\}}$$

with $\max\left\| d_{ij} - \tilde{d}_{ij} \right\| \leq \varepsilon$

Very difficult !

Reduced problem can be solved explicitly

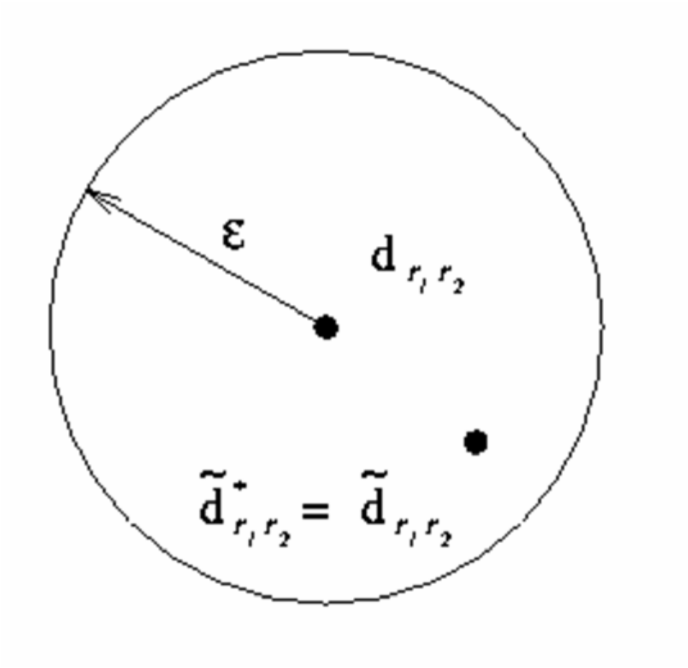
$$\Pi = \iint_A \tilde{X}_{uu}^2 + 2\cdot\tilde{X}_{uv}^2 + \tilde{X}_{vv}^2 \, du\, dv \rightarrow \min_{\tilde{d}_{r_1 r_2}}$$

with $\left\| d_{r_1 r_2} - \tilde{d}_{r_1 r_2} \right\| \leq \varepsilon$

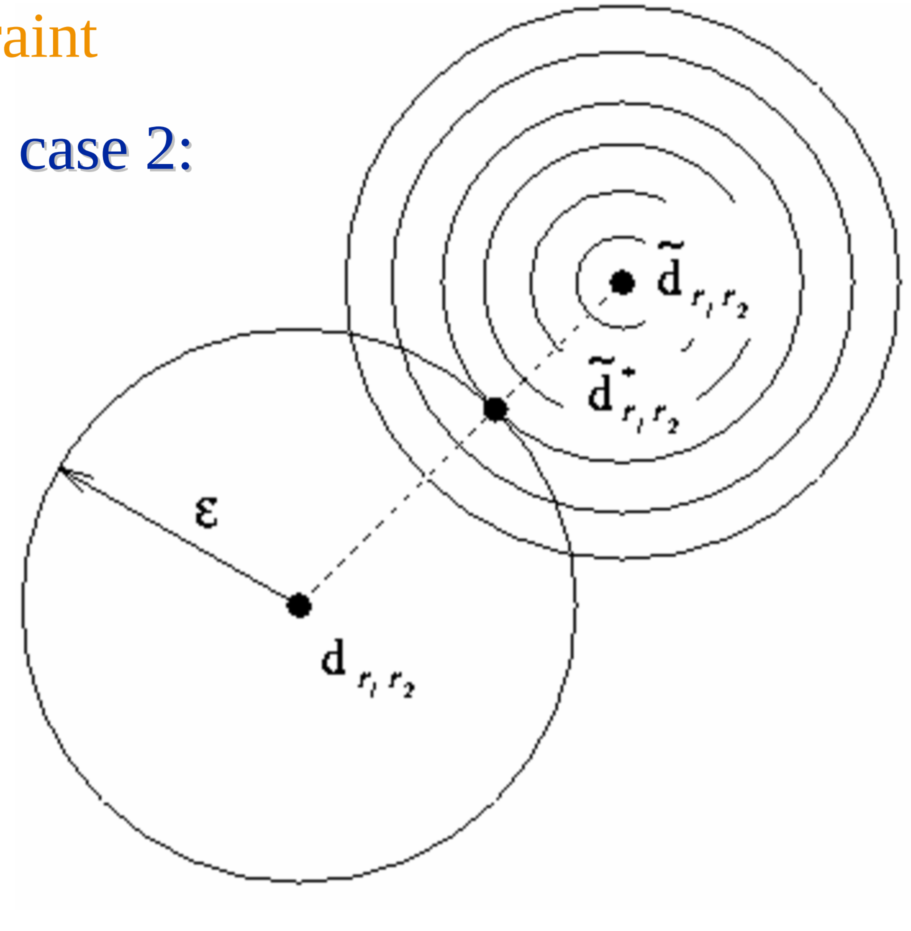
Solution

$\tilde{d}_{r_1 r_2}$ is solution without contraint

case 1:



case 2:

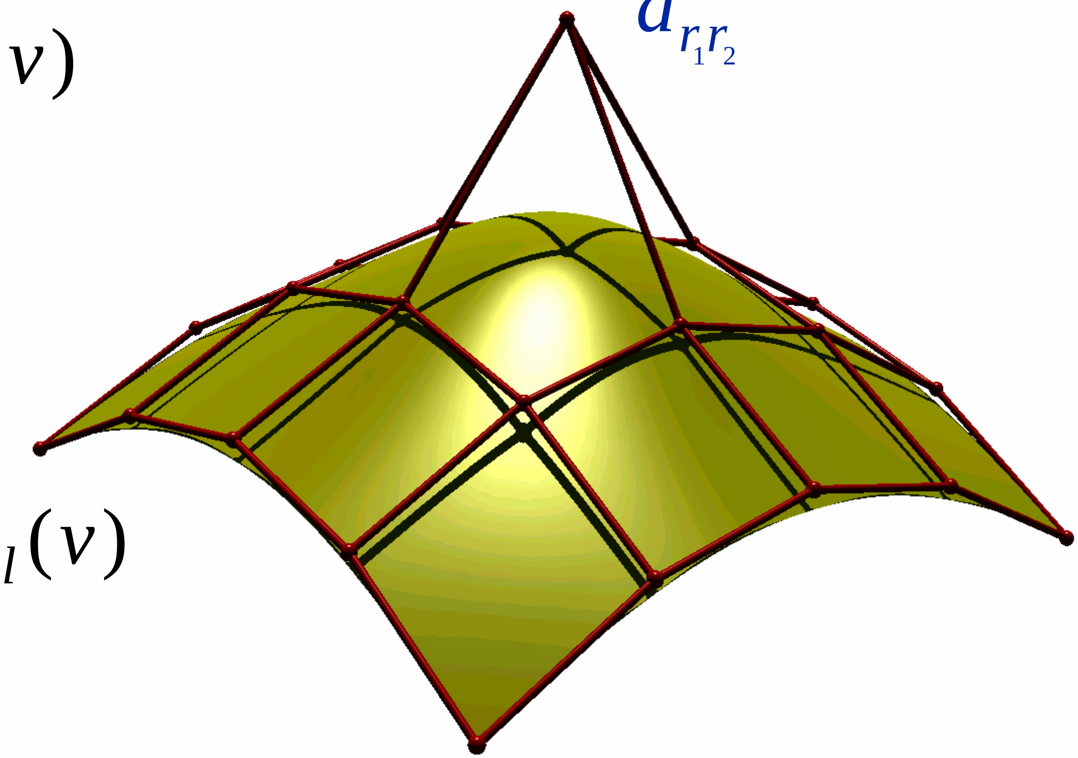


One control point (one step)

$$X = \sum_{i=0}^{n} \sum_{j=0}^{m} d_{ij} \cdot N_{i,k}(u) \cdot N_{j,l}(v)$$

change $d_{r_1 r_2}$

$$\tilde{X} = \sum_{\substack{i=0 \\ (i,j)\neq(r_1,r_2)}}^{n} \sum_{j=0}^{m} d_{ij} \cdot N_{i,k}(u) \cdot N_{j,l}(v) + \tilde{d}_{r_1 r_2} \cdot N_{r_1,k}(u) \cdot N_{r_2,l}(v)$$



The new control point (1)

$$\frac{\partial \Pi\left(\tilde{d}_{r_1 r_2}\right)}{\partial \tilde{d}_{r_1 r_2}} \overset{!}{=} 0$$

unique minimum →

$$\tilde{d}_{r_1 r_2} = \sum_{i=i_0}^{i_1} \sum_{\substack{j=j_0 \\ (i,j)\neq(r_1,r_2)}}^{j_1} \gamma_{ij} \cdot d_{ij}$$

$$i_0 = \max\{0, r_1 - k + 1\} \quad i_1 = \min\{0, r_1 + k - 1\}$$

$$j_0 = \max\{0, r_2 - l + 1\} \quad j_1 = \min\{0, r_2 + l - 1\}$$

The new control point (2)

$$\gamma_{ij} = -\frac{U_{i,r_1}^{2,2}\cdot V_{j,r_2}^{0,0} + 2\cdot U_{i,r_1}^{1,1}\cdot V_{j,r_2}^{1,1} + U_{i,r_1}^{0,0}\cdot V_{j,r_2}^{2,2}}{U_{r_1,r_1}^{2,2}\cdot V_{r_2,r_2}^{0,0} + 2\cdot U_{r_1,r_1}^{1,1}\cdot V_{r_2,r_2}^{1,1} + U_{r_1,r_1}^{0,0}\cdot V_{r_2,r_2}^{2,2}}$$

$$U_{i,r_1}^{p,q} = \int_{u_0}^{u_1} N_{i,k}^{(p)}(u)\cdot N_{r_1,k}^{(q)}(u)\,du$$

$$V_{j,r_{21}}^{p,q} = \int_{v_0}^{v_1} N_{j,l}^{(p)}(v)\cdot N_{r_2,l}^{(q)}(v)\,dv$$

$$\sum_{\substack{i=i_0 \\ (i,j)\neq(r_1,r_2)}}^{i_1}\sum_{j=j_0}^{j_1} \gamma_{ij} = 1$$

Ranking number

- Which control point should be changed ?

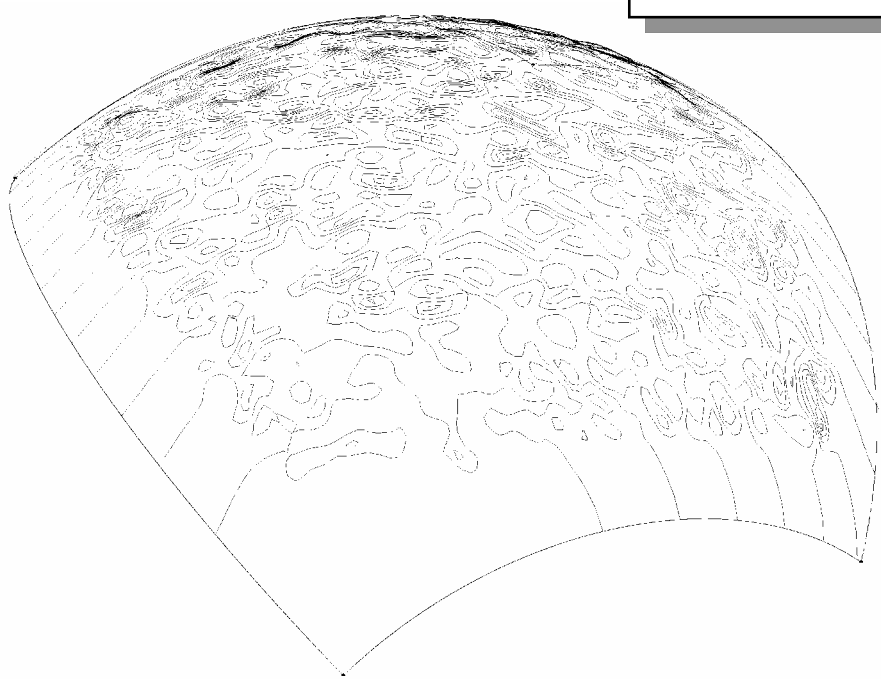
$$z_{r_1 r_2} = \Pi\left(d_{r_1 r_2}\right) - \Pi\left(\tilde{d}_{r_1 r_2}\right)$$

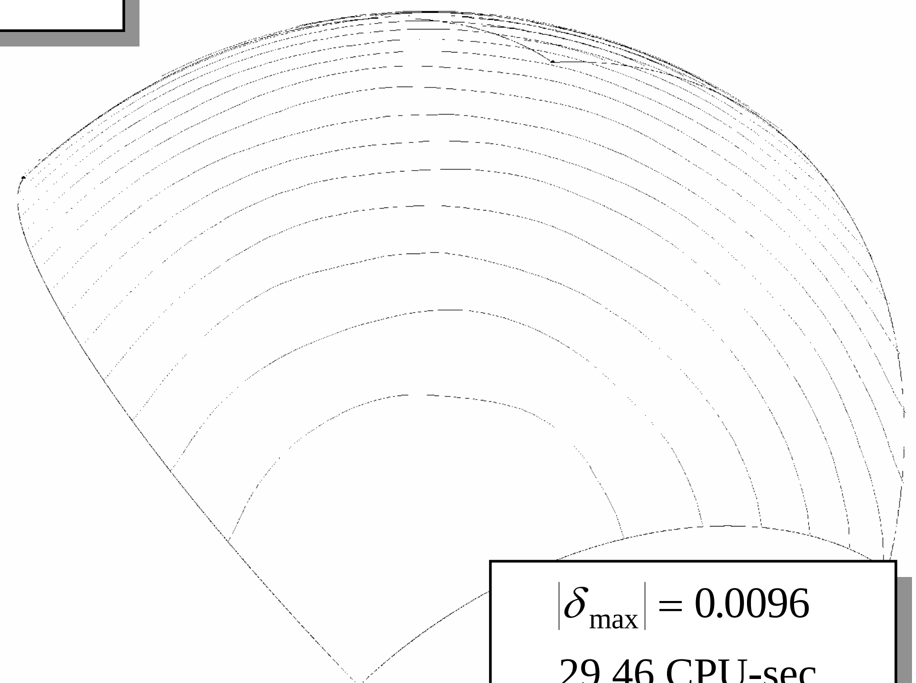
$$\alpha_1 \le r_1 \le n - \beta_1$$
$$\alpha_2 \le r_2 \le m - \beta_2$$

$$z_{r_1, r_2} = \left(d_{r_1 r_2} - \tilde{d}_{r_1 r_2}\right) \cdot \left(U^{2,2}_{r_1, r_1} \cdot V^{0,0}_{r_2, r_2} + 2 \cdot U^{1,1}_{r_1, r_1} \cdot V^{1,1}_{r_2, r_2} + U^{0,0}_{r_1, r_1} \cdot V^{2,2}_{r_2, r_2}\right)$$

Examples (1)

52 × 52 control points
quartic B-spline surface
uniform knot-vector
$\varepsilon = 0.02$

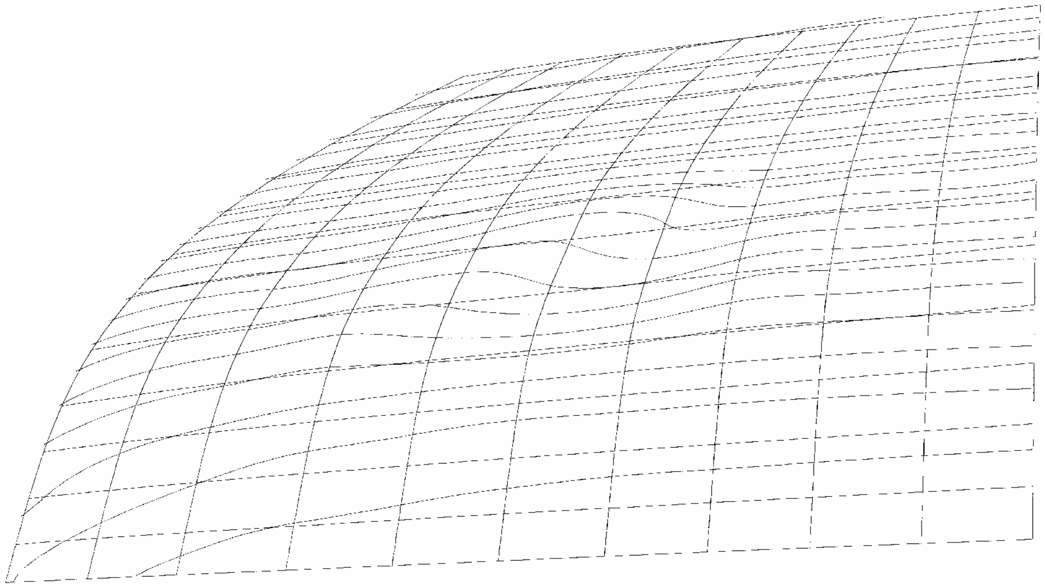




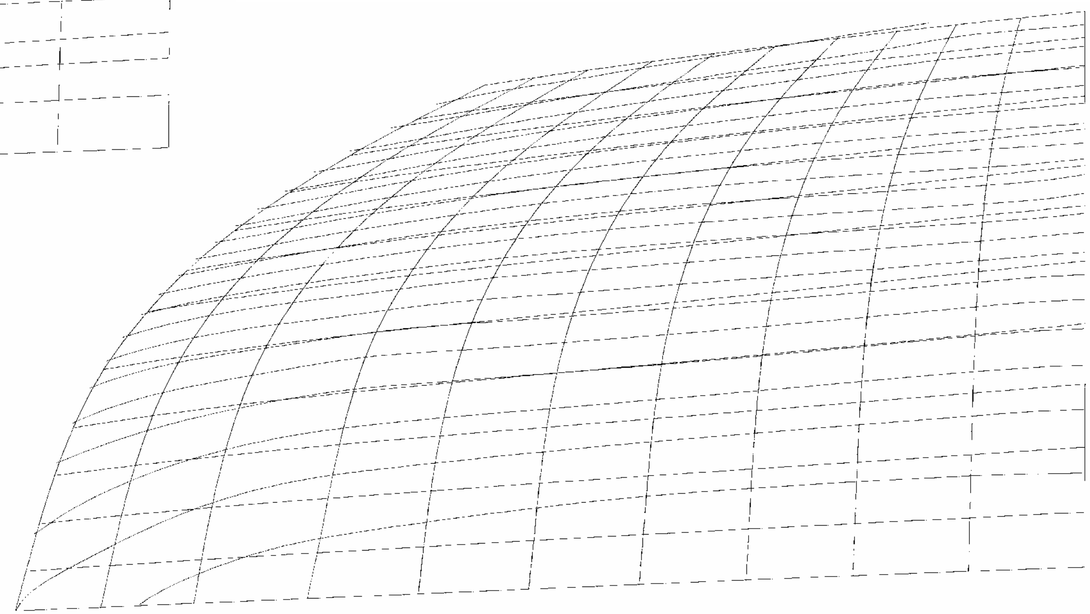
$|\delta_{max}| = 0.0096$

29.46 CPU-sec.
10.000 iterations

Examples (2)



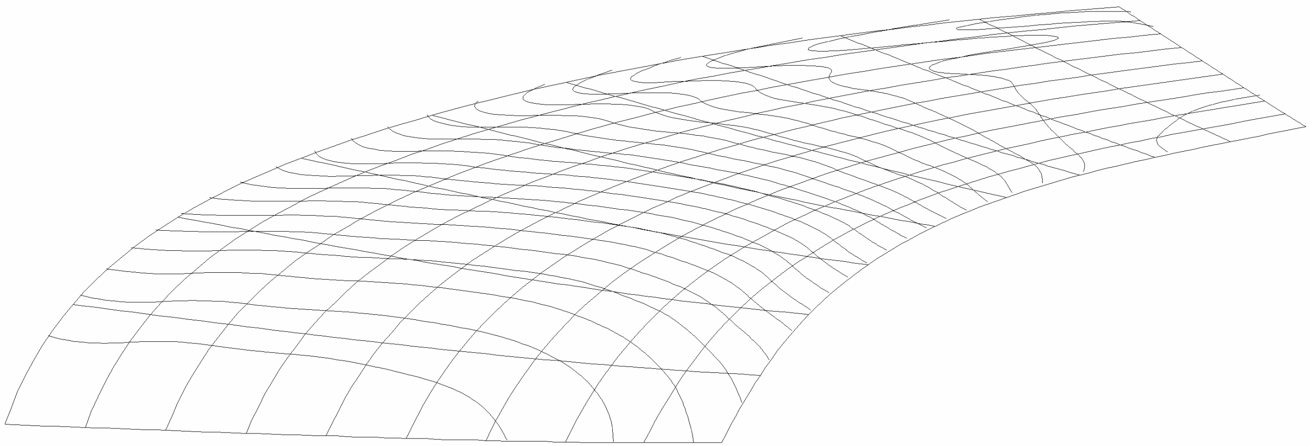
13 × 15 control points
cubic B-spline surface
non-uniform knot-vector
$\varepsilon = 1$



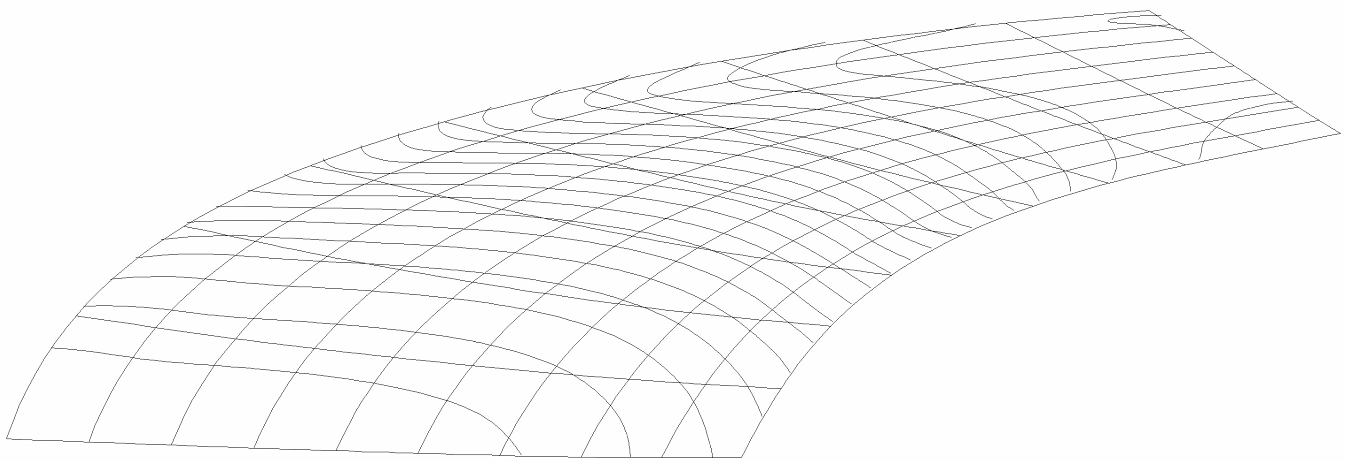
$|\delta_{max}| = 0.27$

0.83 CPU-sec.
2.500 iterations

Examples (3)



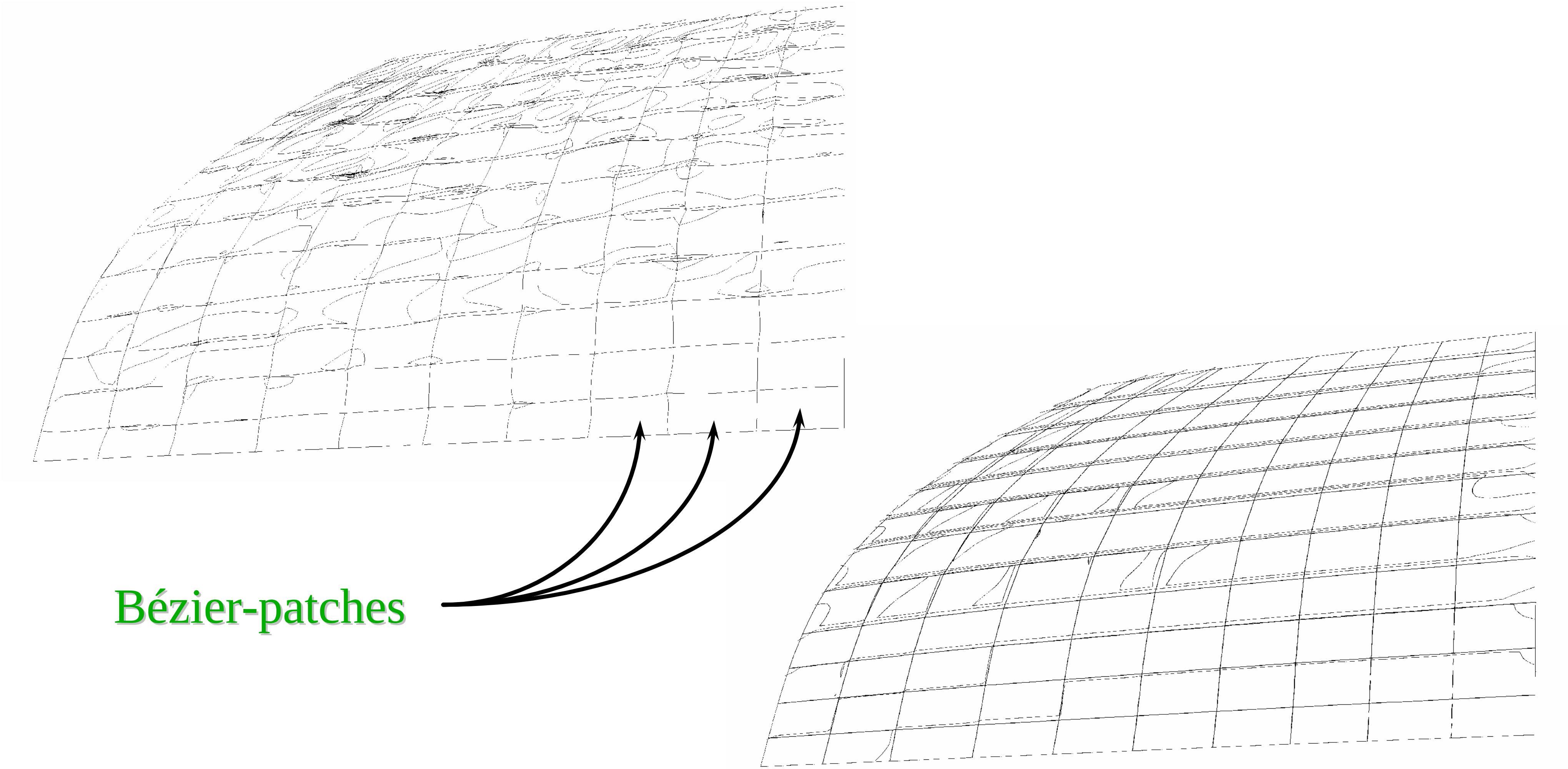
12 × 12 control points
cubic B-spline surface
non-uniform knot-vector
$\varepsilon = 1$



$|\delta_{\max}| = 0.46$

0.7 CPU-sec.
2.500 iterations

B-spline surfaces with multiple inner knots



Extension to Bézier-spline surfaces

① Check for C^1 continuity at the inner knots

② If C^1 remove knot (exact !)

③ Use an extended fairness functional

- Thin-plate + continuity

④ Proceed like before

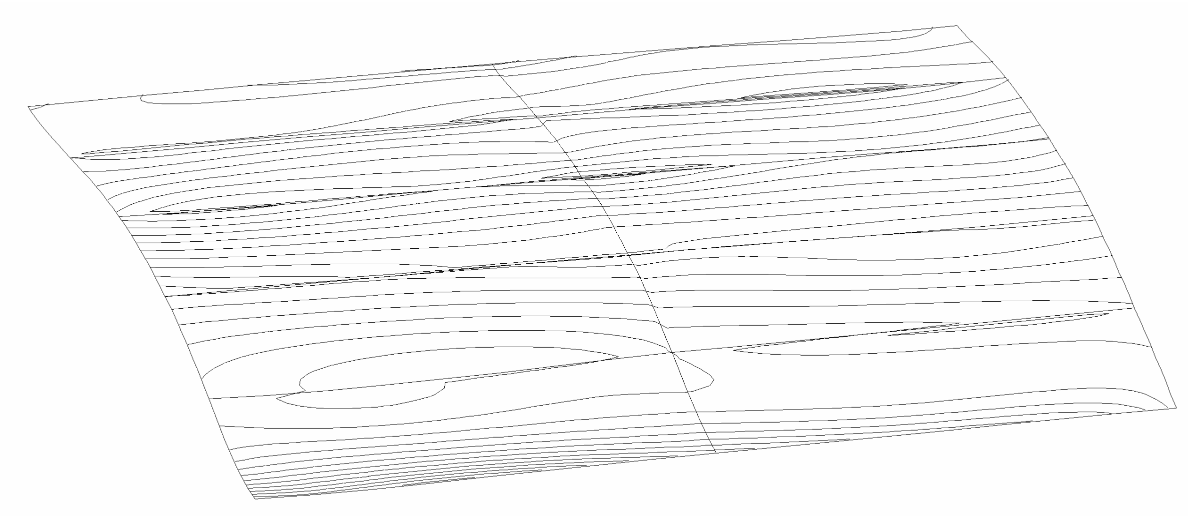
Fairness criterion: Thin-plate energy + continuity

$$\Phi = (1-\lambda)\cdot\iint_A \tilde{X}_{uu}^2 + 2\cdot\tilde{X}_{uv}^2 + \tilde{X}_{vv}^2\,du\,dv$$

$$+\lambda \sum_{\substack{i=k\\ u_i\neq u_{i-1}\\ mult=k-1}}^{m} \sum_{j=l-1}^{n+1} \left(\frac{\partial\,\tilde{X}(u_i^-, v_j)}{\partial\,u} - \frac{\partial\,\tilde{X}(u_i^+, v_j)}{\partial\,u} \right)^2$$

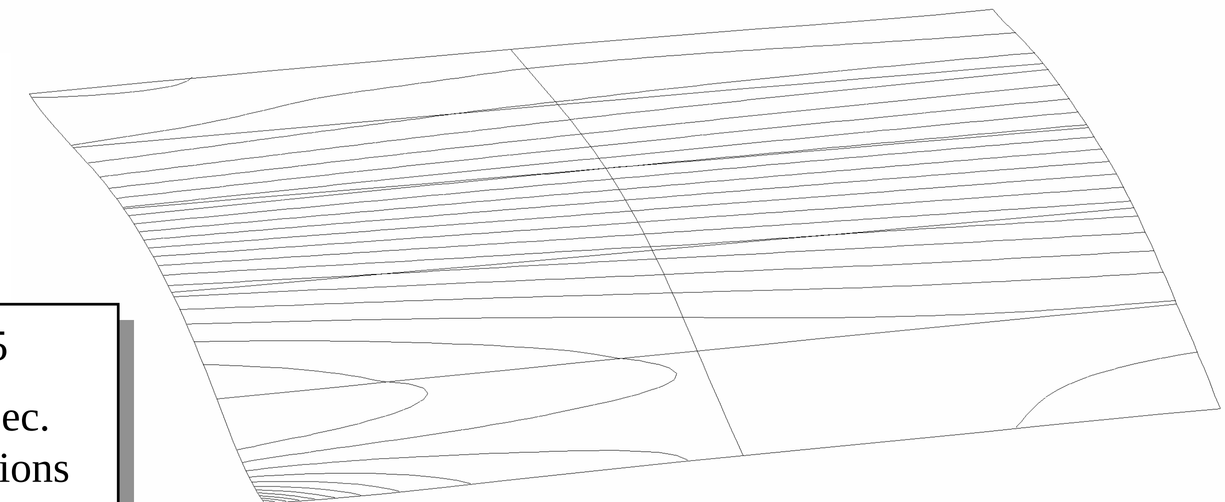
$$+\lambda \sum_{i=k-1}^{m+1} \sum_{\substack{j=l\\ v_i\neq v_{i-1}\\ mult=l-1}}^{n} \left(\frac{\partial\,\tilde{X}(u_i, v_j^-)}{\partial\,v} - \frac{\partial\,\tilde{X}(u_i, v_j^+)}{\partial\,v} \right)^2 \rightarrow \min_{\tilde{d}_{r_1 r_2}}$$

Examples (4)



21 × 11 control points
degree 5 × 6
uniform knot-vector

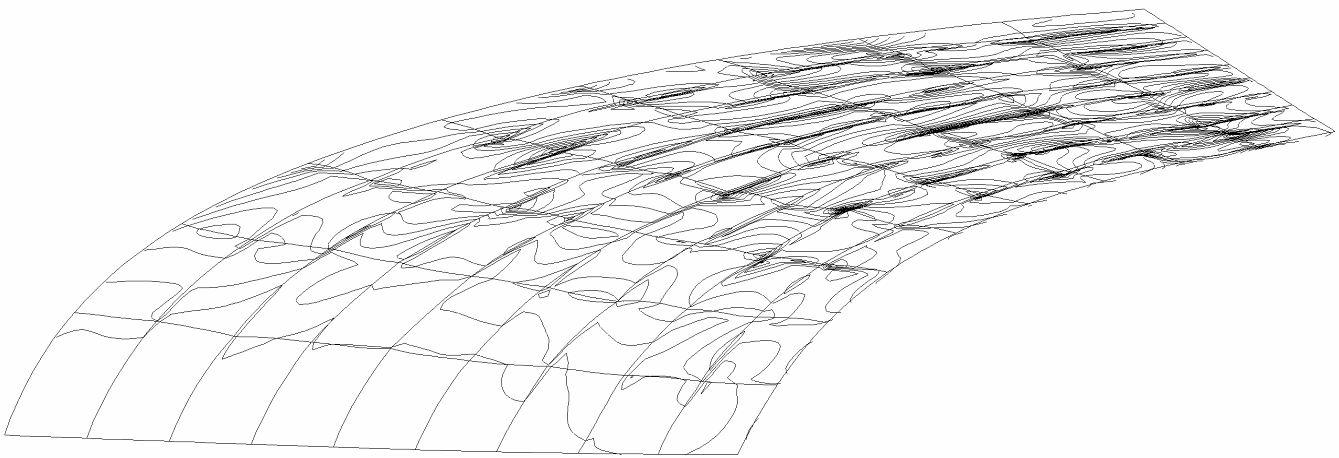
$\varepsilon = 10$
$\lambda = 0.95$



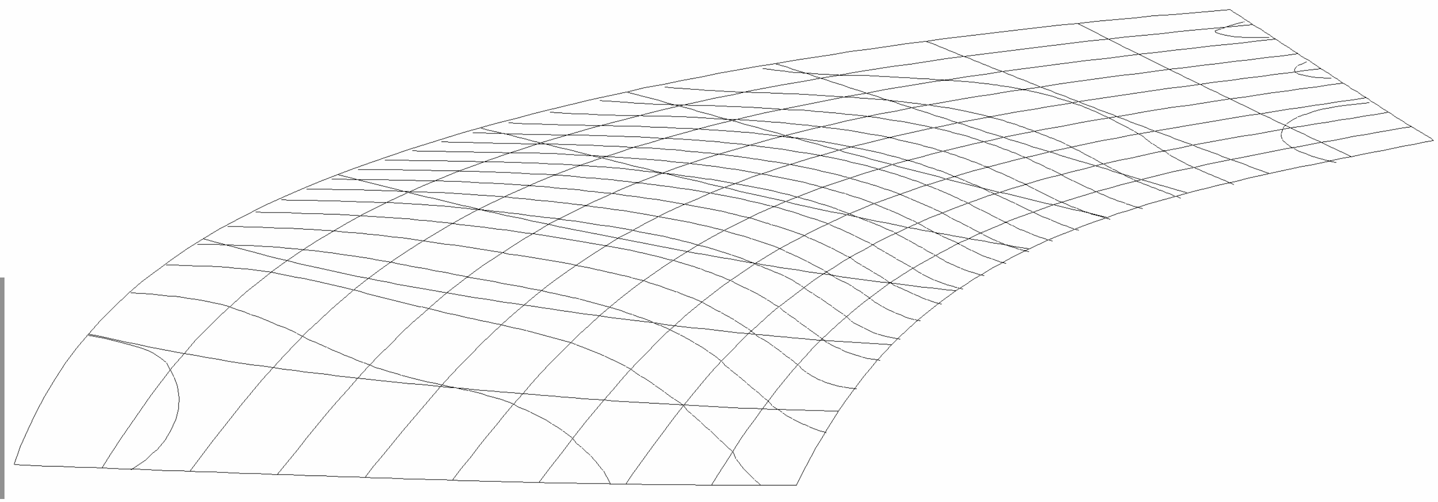
$|\delta_{max}| = 2.015$

14.66 CPU-sec.
250.000 iterations

Examples (5)

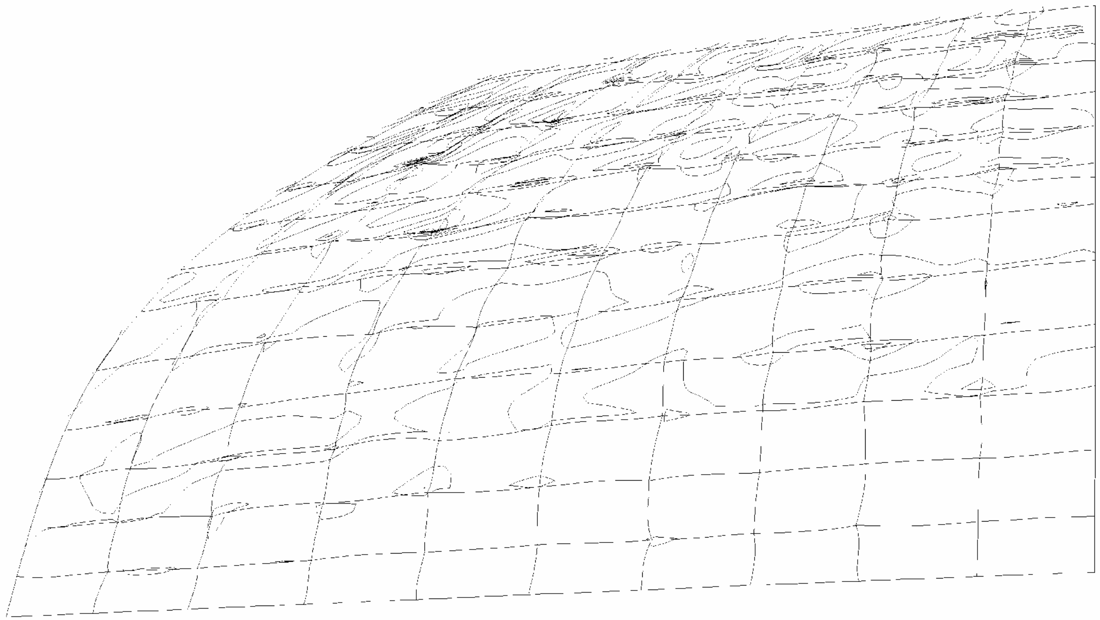


28 × 28 control points
cubic
non-uniform knot-vector
$\varepsilon = 10$
$\lambda = 0.95$

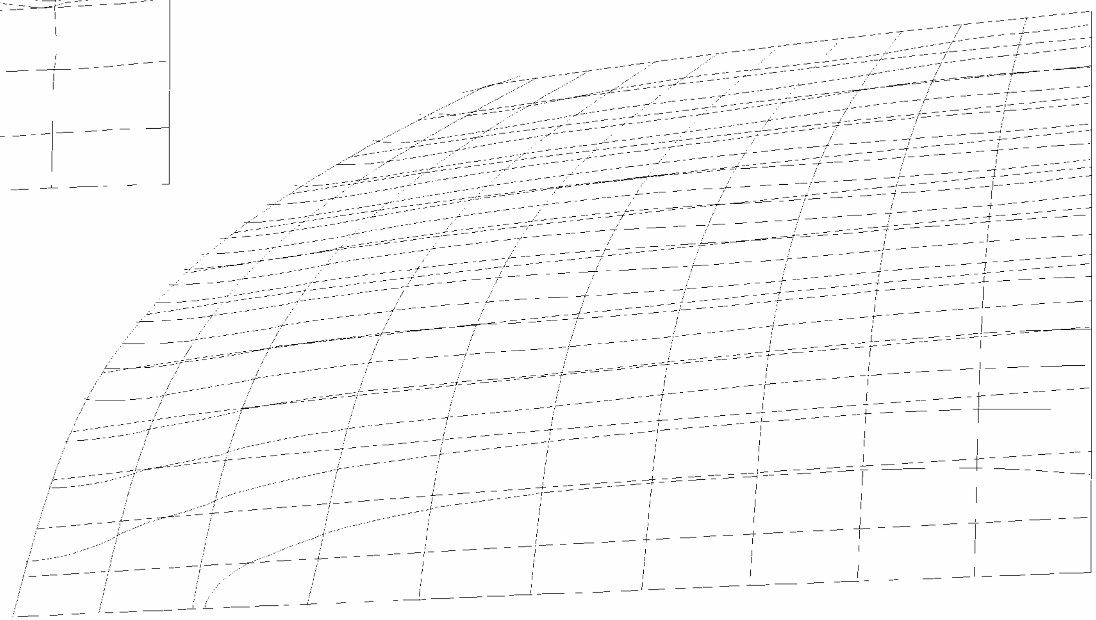


$|\delta_{max}| = 1.6$
489 CPU-sec.
2.500.000 iterations

Examples (6)



13 × 14 control points
cubic
non-uniform knot-vector
$\varepsilon = 10$
$\lambda = 0.85$



$|\delta_{max}| = 0.773$
70.1 CPU-sec.
250.000 iterations

Conclusion

- explicit solutions (in every step)
- local scheme
- fulfilling a distance tolerance
- fast